Int J Theor Phys (2010) 49: 1288–1301
DOI 10.1007/s10773-010-0310-9

Unified Field Theoretical Models from Generalized Affine Geometries

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Received: 7 March 2010 / Accepted: 10 March 2010 / Published online: 24 March 2010

Abstract New model of a non-dualistic Unified Theory is analyzed. This model is based in a manifold equipped with an underlying hypercomplex structure and zero non-metricity, that makes it geometricaly and physically consistent. Wormhole solution from this new model is presented and is explicitly compared with our previous one coming from the Einstein-Non Abelian Born-Infeld theory (in Class. Quantum Gravity 22:4987–5004, 2005). We find that the torsion plays in this unified theory similar role that Yang Mills type strength field coming from the non-Abelian Born-Infeld energy momentum tensor. The meaning of the Yang-Mills ansatz based in the alignment of the isospin with the frame geometry of the spacetime is discussed.

Keywords Unified theories · Gravitation · Non-Riemmanian geometry

1 Introduction and Summary

From long time ago in the history of the physics the formulation of the gravitational theory together with the other interactions was one of the main points focused by the researcher,and this fact is not extrange: our experience has shown that formerly unrelated parts of physics could be fused into one single conceptual formalism by a new theoretical perspective: electricity and magnetism, optics and electromagnetism, thermodynamics and statistical mechanics, inertial and gravitational forces. In the second half of the 20th century, the electromagnetic and weak nuclear forces have been bound together as an electroweak force;

Work partially suported by CAPES Brazilian funds.

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a powerful scheme was devised to also include the strong interaction (chromodynamics), and led to the standard model of elementary particle physics. Unification with the fourth fundamental interaction, gravitation, is in the focus of much present research in classical general relativity, supergravity, superstring, and supermembrane theory but has not yet met with success [22]. The question is what we miss in this task.

As is well known, spin-angular momentum and mass appear in very symmetric way in non-gravitational physics. Moreover, the labels of the irreducible representations of the Poincare group [1] are precisely the mass and the spin. Then, in view of this fact, one are able to note that the Einstein theory is incomplete because only energy-momentum and not spin-angular momentum is given dynamical importance for the structure (geometrical properties) of the space-time.

The Einstein theory is deduced assuming *a priori* the Riemannian structure of the space-time, that is without torsion. Arguments have been given that the spacetime should exhibit both curvature and torsion in the presence of the matter [2–6].

The coupling of spin density to torsion of spacetime is natural when the R_4 geometry is extended to U_4, from a Riemannian to Riemannian-Cartan geometry [2–4, 6]. For instance, the Einstein-Cartan theory is the simplest generalization of the Einstein's theory obtained in the U_4 geometry. But, however, in the usual Einstein-Cartan geometry [2–4, 6] the spin-geometry coupling and the energy-geometry coupling still appears. The Christoffel connection depends upon the metric and its derivatives, but the torsion terms are regarded as independent fields. Then, the direct consequence that we have upon variation with respect to the metric and the contorsion, second order differential equations for $g_{\mu\nu}$ and algebraic equations for $T_{\mu\nu\rho}$. This fact is unnatural and its meaning is obscure, indeed that we can eliminate the torsion of the field equations and obtain an Einstein theory with a modificated matter field Lagrangian. Thus, the theories involved are dynamically equivalent [7].

At this stage one suspect that a more deep question is involved in the same root of the problem: spin, energy-matter and spacetime structure. The theories described above, besides the obvious difference of the spin-torsion coupling, is that both Einstein and Einstein-Cartan are dualistic theories: we must to include the fields (matter) by mean the addition of a (non-geometric) Lagrangian to the gravitational (geometrical) one. Einstein himself pointed out this fact as "undesirable" and only has the status of some bridge towards the final unified theory. It seems reasonable, for instant, to continue these efforts in order to obtain the correct way to solve the important problem of the natural unification of the natural world (matter, energy, spin).

In this report we present a new model of a non-dualistic Unified Theory. This model is absolutely consistent from the mathematical and geometrical point of view and is based in a manifold equipped with an underlying hypercomplex structure and zero non-metricity, that lead the important fact that the Torsion of the spacetime structure turns to be totally antisymmetric: this is the only important case that this type of affine geometrical frameworks are compatible with the physical "equivalence principle". Also we shown that interesting wormhole solutions, similarly to the previous reference with the non Abelian Born-Infeld theory, can be obtained in this theory. The solution of this model is explicitly compared with our previous one and we find that the torsion plays in this unified theory similar role that Yang Mills type strength field coming from the non-Abelian Born-Infeld energy momentum tensor of our previous reference. The another important result is that the meaning of the Hosoya-Ogura ansatz (namely, the alignment of the isospin with the frame geometry of the spacetime) is completely elucidated.

2 The Spacetime Manifold and the Geometrical Action

The starting point is an hypercomplex construction of the (metric compatible) space-time manifold [8]. The main ingredients for this construction are:

The metric

$$g_{\mu\nu} = \overline{g}_{\mu\nu} = g_{\nu\mu} \in \mathbb{R} \quad \text{with } \nabla g = 0 \tag{1}$$

Also, we assume that the potential torsion exists and arises in a natural form considering that the geometry is reductive (the ∇ for the covariant derivative with respect the full connection Γ). This potential torsion has the following properties

$$\begin{aligned} f_{\mu\nu} &= \overline{f}_{\mu\nu} = -f_{\nu\mu} \in \mathbb{HC} \\ \nabla_{[\rho} f_{\mu\nu]} &= T_{\mu\nu\rho} \\ &= \varepsilon_{\mu\nu\rho\sigma} h^{\sigma} \end{aligned} \tag{2}$$

where the last equality coming from the full antisymmetry of the Torsion field. Immediately we can see, as a consequence of the above statements, the following

(i) the torsion is the dual of an axial vector h^{σ};
(ii) from (i), the existence in the spacetime of a completely antisymmetric tensor covariantly constant $\varepsilon_{\mu\nu\rho\sigma}$ $(\nabla\varepsilon = 0)$.

Notice that, as we will show in detail elsewhere [9], the choice for the real nature of the metric and the pure hypercomplex potential tensor coming from the Hermitian nature of the theory: if we assume (1), the condition (2) arises automatically.

The second important point is to consider [10, 11] the extended curvature

$$\mathcal{R}^{ab}_{\mu\nu} = R^{ab}_{\mu\nu} + \Sigma^{ab}_{\mu\nu} \tag{3}$$

with

$$\begin{aligned} R^{ab}_{\mu\nu} &= \partial_{\mu}\omega^{ab}_{\nu} - \partial_{\nu}\omega^{ab}_{\mu} + \omega^{ac}_{\mu}\omega_{\nu c}{}^{b} - \omega^{ac}_{\nu}\omega_{\mu c}{}^{b} \\ \Sigma^{ab}_{\mu\nu} &= -\left(e^{a}_{\mu}e^{b}_{\nu} - e^{a}_{\nu}e^{b}_{\mu}\right) \end{aligned}$$

We assume here ω^{ab}_{ν} a $SO(d-1,1)$ connection and e^{a}_{μ} is a vierbein field. The (3) can be obtained, for example, using the formulation that was pioneering introduced in seminal works by E. Cartan long time ago [10, 11]. Is well known that in such an formalism the gravitational field is represented as a connection one form associated with some group which contains the Lorentz group as subgroup. The typical example is provided by the $SO(d,1)$ de Sitter gauge theory of gravity. In this specific case, the $SO(d,1)$ the gravitational gauge field $\omega^{AB}_{\mu} = -\omega^{BA}_{\mu}$ is broken into the $SO(d-1,1)$ connection ω^{ab}_{μ} and the $\omega^{da}_{\mu} = e^{a}_{\mu}$ vierbein field, with the dimension d fixed. Then, the de Sitter (anti-de Sitter) curvature

$$\mathcal{R}^{ab}_{\mu\nu} = \partial_{\mu}\omega^{AB}_{\nu} - \partial_{\nu}\omega^{AB}_{\mu} + \omega^{AC}_{\mu}\omega_{\nu C}{}^{B} - \omega^{AC}_{\nu}\omega_{\mu C}{}^{B} \tag{4}$$

splits in the curvature (3).

Now we define the following geometrical object

$$\mathcal{R}^{a}{}_{\mu} = \lambda\left(e^{a}{}_{\mu} + f^{a}{}_{\mu}\right) + R^{a}{}_{\mu} \qquad \left(M^{a}{}_{\mu} \equiv e^{a\nu} M_{\nu\mu}\right) \tag{5}$$

The action will contains, as usual, $\mathcal{R} = \det(\mathcal{R}^{a}{}_{\mu})$ as the geometrical object that defines the dynamics of the theory. The particularly convenient definition of $\mathcal{R}^{a}_{\mu}$ makes easy to establish the equivalent expression in the spirit of the Unified theories developed time ago by Eddington, Einstein and Born and Infeld for example:

$$\sqrt{\det \mathcal{R}^{a}{}_{\mu}\mathcal{R}_{a\nu}} = \sqrt{\det\left[\lambda^{2}\left(g_{\mu\nu} + f^{a}{}_{\mu} f_{a\nu}\right) + 2\lambda R_{(\mu\nu)} + 2\lambda f^{a}{}_{\mu} R_{[a\nu]} + R^{a}{}_{\mu} R_{a\nu}\right]} \tag{6}$$

where $R_{\mu\nu} = R_{(\mu\nu)} + R_{[\mu\nu]}$.

The important point to consider in this simple Cartan inspired model is that, although a cosmological constant λ is required, the expansion of the action in four dimensions lead automatically the Hilbert-Einstein part when $f^{a}{}_{\mu} = 0$. Explicitly ($R = g^{\alpha\beta} R_{\alpha\beta}$)

$$\begin{aligned} S = \int d^{4}x(e+f)\Big\{ & \lambda^{4} + \lambda^{3}(R + f^{a}{}_{\mu} R^{\mu}{}_{a}) \\ & + \frac{\lambda^{2}}{2!}\left[R^{2} - R^{\mu\nu}R_{\mu\nu} + \left(f^{a}{}_{\mu}R^{\mu}{}_{a}\right)^{2} - f^{\mu\nu} f^{\rho\sigma} R_{\mu\rho} R_{\nu\sigma}\right] \\ & + \frac{\lambda}{3!}\left[R^{3} - 3RR^{\mu\nu}R_{\mu\nu} + 2R^{\mu\alpha}R_{\alpha\beta}R^{\beta}{}_{\mu} + \left(f^{a}{}_{\mu}R^{\mu}{}_{a}\right)^{3} \right. \\ & \left. - 3\left(f^{a}{}_{\mu}R^{\mu}{}_{a}\right) f^{\mu\nu} f^{\rho\sigma} R_{\mu\rho}R_{\nu\sigma} + 2 f^{\mu\nu} R_{\mu}{}^{\alpha} R_{\alpha\beta} R^{\beta}{}_{\nu}\right] + \det(R_{\mu\nu})\Big\} \end{aligned} \tag{7}$$

3 The Dynamical Equations

Defining

$$\eta_{ab}\mathcal{R}^{a}{}_{\mu}\mathcal{R}^{b}{}_{\nu} \equiv G_{\mu\nu} \tag{8}$$

the variation with respect to the metric $g_{\mu\nu}$ is straightforward

$$\frac{\delta\sqrt{G}}{\delta g^{\alpha\beta}} = \frac{\sqrt{G}}{2}\left(G^{-1}\right)^{\mu\nu}\left[\lambda^{2}\left(-g_{\beta\nu}g_{\alpha\mu} + f_{\beta\nu}f_{\alpha\mu}\right) + 2\lambda f_{\alpha\mu}R_{[\beta\nu]}\right] \tag{9}$$

In order to compute the variation with respect to f it is useful to remind the structure of the Riemann tensor [13]

$$R_{\mu\nu} = \overbrace{\overset{\circ}{R}_{\mu\nu} - T_{\mu\rho}{}^{\alpha} T_{\alpha\nu}{}^{\rho}}^{R_{(\mu\nu)}} + \overbrace{\overset{\circ}{\nabla}_{\alpha} T_{\mu\nu}{}^{\alpha}}^{R_{[\mu\nu]}} \tag{10}$$

where $\overset{\circ}{R}_{\mu\nu}$ and $\overset{\circ}{\nabla}_{\alpha}$ are the Riemann tensor and the covariant derivative computed from the Christoffel symbol $\{^{\rho}_{\mu\nu}\}$. Then, using the last expression (10), we obtain for the f variation

$$\frac{\delta\sqrt{G}}{\delta f_{\sigma\tau}} = \nabla_{\rho}\left(\frac{\partial\sqrt{G}}{\partial T_{\rho\sigma\tau}}\right) - \frac{\partial\sqrt{G}}{\partial f_{\sigma\tau}} \equiv \nabla_{\rho}\mathbb{T}^{\rho\sigma\tau} - \mathbb{F}^{\sigma\tau} = 0 \tag{11}$$

From above expressions is not difficult to see that the full set of equations involved in our task are

$$R_{\mu\nu} = -2\lambda\left(g_{\mu\nu} + f_{\mu\nu}\right) \tag{12}$$

$$\nabla_\rho \left(\frac{\partial\sqrt{G}}{\partial T_{\rho\sigma\tau}}\right) - \frac{\partial\sqrt{G}}{\partial f_{\sigma\tau}} \equiv \nabla_\rho \mathbb{T}^{\rho\sigma\tau} - \mathbb{F}^{\sigma\tau} = 0 \tag{13}$$

4 The Dynamical Equations II: Physical and Geometrical Interpretation

The variational equations (in the Palatini's sense [10, 11, 13]) (12) and (13) above, despite their simplest and compact form, it is necessary to find what is the deep physical and geometrical meaning inside they.

For expression (13) we have a highly nonlinear dynamical (propagating) equation for the torsion field, where the variation was performed with respect to their potential $f_{\mu\nu}$ and having a nonlinear term proportional to $f_{\mu\nu}$ playing the role of current for the $\mathbb{T}^{\rho\sigma\tau}$. Then, the potential two form is associated nonlinearly to the torsion field as his source regarding similar association between the electromagnetic field and the spin in particle physics.

For the expression (12), firstly is useful to split the equation into the symmetric and the antisymmetric parts using (10)

$$R_{(\mu\nu)} = \overset{\circ}{R}_{\mu\nu} - T_{\mu\rho}{}^{\alpha} T_{\alpha\nu}{}^{\rho} = -2\lambda g_{\mu\nu} \tag{14}$$

$$\begin{aligned} R_{[\mu\nu]} &= \overset{\circ}{\nabla}_\alpha T^{\alpha}{}_{\mu\nu} = -2\lambda f_{\mu\nu} \\ &= \nabla_\alpha T^{\alpha}{}_{\mu\nu} \end{aligned} \tag{15}$$

the last equality coming from (2). The symmetric part (14) can be written in a "GR" suggestive fashion

$$\overset{\circ}{R}_{\mu\nu} = -2\lambda g_{\mu\nu} + T_{\mu\rho}{}^{\alpha} T_{\alpha\nu}{}^{\rho} \tag{16}$$

we can advertise that the equation has the aspect of the Einstein equations with the cosmological term modified by the torsion symmetric term $T_{\mu\rho}{}^{\alpha} T_{\alpha\nu}{}^{\rho}$. This can be interpreted by the energy of the gravitational field itself.

The second antisymmetric part (15) is more involved. In order to understand it, will be necessary use the language of differential forms to rewrite they that, beside their symbolic and conceptual simplicity, permit us to check consistency and covariance step by step.

$$\begin{aligned} \nabla_\alpha T^{\alpha}{}_{\mu\nu} &= -2\lambda f_{\mu\nu} \\ d^*T &= -2\lambda\, {}^*f \end{aligned} \tag{17}$$

now, using (2) ($T = {}^*h$)

$$dh = -2\lambda\, {}^*f \quad \Rightarrow \quad {}^*f = -\frac{1}{2\lambda}\, dh \tag{18}$$

in more familiar form

$$\nabla_\mu h_\nu - \nabla_\nu h_\mu = -2\lambda\, {}^*f_{\mu\nu} \tag{19}$$

then follows using (2) again: $T = df = {}^*h$ and (17)

$$d\,{}^*f = 0 \tag{20}$$

and fundamentally

$$df = -\frac{1}{2\lambda}\, d^*dh = T = {}^*h \tag{21}$$

$$d^*dh = -2\lambda\, {}^*h \tag{22}$$

that we can recognize the Laplace-de Rham operator that help us to write the wave covariant equation

$$\begin{aligned} \left[(d\delta + \delta d) + 2\lambda\right] {}^*h &= 0 \\ (\Delta + 2\lambda)\, {}^*h &= 0 \end{aligned} \tag{23}$$

If we start with the potential is not difficult to see that equivalent equation can be find

$$(\Delta + 2\lambda)\, {}^*f = 0 \tag{24}$$

Notice that (23) coming from (18) and is consequence of the Tfh-relation ($T = df = {}^*h$) but (24) comes directly from (17). The geometric interplay is[1]

$$\begin{tikzcd} & \boxed{T} \arrow[dl, shift right, "\int"'] \arrow[dr, shift left, "(-1)^{d+1}*"] & \\ \boxed{f} \arrow[ur, shift right, "d"'] \arrow[rr, shift right, "-2\lambda \int^*"'] & & \boxed{h} \arrow[ul, shift left, "*"] \arrow[ll, shift right, "-1^* d/2\lambda"'] \end{tikzcd} \tag{25}$$

And finally, the explicit computation of the determinant in ($d = 4$) of expression (8) that will help us in order to compare the unitarian model introduced here (in the sense of Eddington [14, 15]) with the dualistic non Abelian Born-Infeld model of [16], takes the familiar form [16]

$$S = \frac{b^2}{4\pi} \int \sqrt{-g}\, dx^4 \left\{ \overbrace{\sqrt{\gamma^4 - \frac{\gamma^2}{2}\overline{G}^2 - \frac{\gamma}{3}\overline{G}^3 + \frac{1}{8}\left(\overline{G}^2\right)^2 - \frac{1}{4}\overline{G}^4}}^{\equiv \mathbb{R}} \right\} \tag{26}$$

$$G_{\mu\nu} \equiv \left[\lambda^2 \left(g_{\mu\nu} + f^a{}_\mu f_{a\nu}\right) + 2\lambda R_{(\mu\nu)} + 2\lambda f^a{}_\mu R_{[a\nu]} + R^a{}_\mu R_{a\nu}\right] \tag{27}$$

$$G^\nu_\nu \equiv \left[\lambda^2 \left(d + f_{\mu\nu} f^{\mu\nu}\right) + 2\lambda \left(R_S + R_A\right) + \left(R_S^2 + R_A^2\right)\right] \tag{28}$$

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[1] In this paragraph we consider an even number of dimensions to avoid the sign ambiguities with the action of the Hodge operator (*).

where

$$R_S \equiv g^{\mu\nu} R_{(\mu\nu)} \qquad R_A \equiv f^{\mu\nu} R_{[\mu\nu]} \qquad \gamma \equiv \frac{G^\nu_\nu}{d} \qquad \overline{G}_{\mu\nu} \equiv G_{\mu\nu} - \frac{g_{\mu\nu}}{4} G^\nu_\nu$$
$$\overline{G}^\nu_\rho \overline{G}^\rho_\nu \equiv \overline{G}^2 \qquad \overline{G}^\nu_\lambda \overline{G}^\lambda_\rho \overline{G}^\rho_\nu \equiv \overline{G}^3 \qquad \left(\overline{G}^\nu_\rho \overline{G}^\rho_\nu\right)^2 \equiv \left(\overline{G}^2\right)^2 \qquad \overline{G}^\nu_\mu \overline{G}^\mu_\lambda \overline{G}^\lambda_\rho \overline{G}^\rho_\nu \equiv \overline{G}^4 \tag{29}$$

and the relevant quantities involved into the dynamical equations (12, 13) are

$$\mathbb{F}^{\mu\nu} \equiv \frac{\partial L_G}{\partial f_{\mu\nu}} = \frac{\lambda^2 N^{\mu\nu} (\delta^\sigma_\mu \, f^\rho{}_\nu + \delta^\sigma_\nu f^\rho{}_\mu)}{2\mathbb{R}} \tag{30}$$

$$\mathbb{T}^{\epsilon\gamma\delta} \equiv \frac{\partial L_G}{\partial T_{\epsilon\gamma\delta}} = \frac{N^{\mu\nu} M^{\epsilon\cdot\gamma\cdot\delta}_{\cdot\alpha\cdot\beta} (2\lambda \delta^\alpha{}_\mu \delta^\beta{}_\nu + R^\alpha{}_\nu \delta^\beta{}_\mu + R^\alpha{}_\mu \delta^\beta{}_\nu)}{2\mathbb{R}} \tag{31}$$

$$N^{\mu\nu} = g\left[-\gamma^2 G^{\mu\nu} - \gamma (G^2)^{\mu\nu} + \frac{(G^2)^\mu_\mu G^{\mu\nu}}{2} - (G^3)^{\mu\nu} + \frac{4\gamma^3 g^{\mu\nu}}{d} - \frac{\gamma (G^2)^\mu_\mu g^{\mu\nu}}{d} - \frac{(G^3)^\mu_\mu g^{\mu\nu}}{3d}\right] \tag{32}$$

$$M^{\epsilon\cdot\gamma\cdot\delta}_{\cdot\alpha\cdot\beta} = \left(\delta^\epsilon{}_\mu T^{\delta\gamma}{}_\nu + T_\mu{}^{\delta\epsilon} \delta^\gamma{}_\nu\right) \tag{33}$$

5 Exact Solutions in the New UFT Theory: The Wormhole-Instanton

The main motivation in this section is clear: we must equip our "theoretical arena" by studying wormhole solutions beyond to Einstein equations coupled to possible matter fields. We know the that many problems appear in the conventional "dualistic" approach even at the classical level, that make that the "dream" of a quantum formulation of the gravity that permit its interaction with other fields becomes practically impossible. Then, let us construct wormhole solutions in the viewpoint of the UFT model introduced here. The action in four dimensions is given by

$$S = -\frac{1}{16\pi G} \int d^4x \sqrt{\det |G_{\mu\nu}|} \tag{34}$$

$$\mathbb{R} \equiv \sqrt{\gamma^4 - \frac{\gamma^2}{2} \overline{G}^2 - \frac{\gamma}{3} \overline{G}^3 + \frac{1}{8} \left(\overline{G}^2\right)^2 - \frac{1}{4} \overline{G}^4} \tag{35}$$

Scalar curvature R and the torsion 2-form field $T^a_{\mu\nu}$ with a $SU(2)$-Yang-Mills structure are defined in terms of the affine connection $\Gamma^\lambda_{\mu\nu}$ and the $SU(2)$ potential torsion $f^a{}_\mu$ by

$$R = g^{\mu\nu} R_{\mu\nu} \qquad R_{\mu\nu} = R^\lambda_{\mu\lambda\nu}$$
$$R^\lambda_{\mu\lambda\nu} = \partial_\nu \Gamma^\lambda_{\mu\rho} - \partial_\rho \Gamma^\lambda_{\mu\nu} + \cdots \tag{36}$$
$$T^a{}_{\mu\nu} = \partial_\mu f^a{}_\nu - \partial_\nu f^a{}_\mu + \varepsilon^a_{bc} f^b{}_\mu f^c{}_\nu$$

G and Λ are the Newton gravitational constant and the cosmological constant respectively. Notice the important fact that from the last equation for the Torsion 2-form, the potential $f^a{}_\mu$ must be proportional with the antisymmetric part of the affine connection $\Gamma^\lambda_{\mu\nu}$ as in the Strauss-Einstein UFT. As in the case of Einstein-Yang-Mills systems, for our new UFT model it can be interpreted as a prototype of gauge theories interacting with gravity (e.g. QCD, GUTs, etc.). Upon varying the action (31), we obtain the gravitational "Einstein-Eddington-like" equation

$$R_{\mu\nu} = -2\lambda \left(g_{\mu\nu} + f_{\mu\nu}\right) \tag{37}$$

and the field equation for the torsion two form in differential form

$$d^{*}\mathbb{T}^{a} + \frac{1}{2}\varepsilon^{abc}\left(f_{b} \wedge {}^{*}\mathbb{T}_{c} - {}^{*}\mathbb{T}_{b} \wedge f_{c}\right) = \mathbb{F}^{a} \tag{38}$$

where we define as usual

$$\mathbb{T}^{a}{}_{bc} \equiv \frac{\partial L_{NBI}}{\partial T_{a}{}^{bc}} \qquad \mathbb{F}^{a}{}_{bc} \equiv \frac{\partial L_{NBI}}{\partial F_{a}}$$

we are going to seek for a classical solution of (33) and (34) with the following spherically symmetric ansatz for the metric and gauge connection

$$ds^{2} = d\tau^{2} + a^{2}(\tau)\,\sigma^{i} \otimes \sigma^{i} \equiv d\tau^{2} + e^{i} \otimes e^{i} \tag{39}$$

here τ is the Euclidean time and the dreibein is defined by $e^{i} \equiv a^{2}(\tau)\,\sigma^{i}$.The gauge connection is

$$f^{a} \equiv f^{a}_{\mu} dx^{\mu} = h\sigma^{a} \tag{40}$$

for $a = 1, 2, 3$ and for $a = 0$

$$f^{0} \equiv f^{0}_{\mu} dx^{\mu} = s\sigma^{0} \tag{41}$$

this choice for the potential torsion is the most general and consistent from the physical and mathematical point of view, as we will show soon. The σ^{i} one-form satisfies the $SU(2)$ Maurer-Cartan structure equation

$$d\sigma^{a} + \varepsilon^{a}{}_{bc}\sigma^{b} \wedge \sigma^{c} = 0 \tag{42}$$

Notice that in the ansatz the frame and isospin indexes are identified as for the case with the NBI Lagrangian of [16]. The torsion two-form

$$T^{\gamma} = \frac{1}{2}T^{\gamma}{}_{\mu\nu}\, dx^{\mu} \wedge dx^{\nu} \tag{43}$$

becomes

$$\begin{aligned} T^{a} &= df^{a} + \frac{1}{2}\varepsilon^{a}{}_{bc} f^{b} \wedge f^{c} \\ &= \left(-h + \frac{1}{2}h^{2}\right)\varepsilon^{a}{}_{bc}\,\sigma^{b} \wedge \sigma^{c} \end{aligned} \tag{44}$$

Notice that f^{0} plays no role here because we take simply $ds = 0$ (the $U(1)$ component of $SU(2)$, in principle, does not form part of the space spherical symmetry), and the expression for the torsion is analogous to the non Abelian two form strength field of [16]. Also, it is important to note that, when we goes from the Lorentzian to Euclidean gravitational regime,

$it \to \tau$ and the torsion pass from the field of the *Hypercomplex* to the *Complex* numbers, for invariance reasons (geometrically, multiplication of hypercomplex numbers preserves the (square) Minkowski norm $(x^2 - y^2)$ in the same way that multiplication of complex numbers preserves the (square) Euclidean norm $(x^2 + y^2)$). Inserting T^a from (44) into the dynamical equation (38) we obtain

$$\begin{aligned} d^*\mathbb{T}^a + \frac{1}{2}\varepsilon^{abc}\left(f_b \wedge {}^*\mathbb{T}_c - {}^*\mathbb{T}_b \wedge f_c\right) &= {}^*\mathbb{F}^a \\ (-2h + h^2)(1-h)d\tau \wedge e^b \wedge e^c &= -2\lambda d\tau \wedge e^b \wedge e^c \end{aligned} \tag{45}$$

where

$${}^*\mathbb{T}^a \equiv \frac{\lambda\sqrt{|g|}}{\sqrt{3}} hA(-2h+h^2)d\tau \wedge \frac{e^a}{a^2} \tag{46}$$

$${}^*\mathbb{F}^a = -\frac{2\lambda^2\sqrt{|g|}}{\sqrt{3}} hA \frac{d\tau \wedge e^b \wedge e^c}{a^3} \tag{47}$$

$$A \equiv \lambda^4\left[(1+\alpha)^2 + \alpha/2\right] \tag{48}$$

and

$$\alpha = \frac{1}{2}\left(s^2 + 3h^2\right) \tag{49}$$

from expression (45) we have an algebraic cubic equation for h

$$(-2h + h^2)(1-h) + 2\lambda = 0 \tag{50}$$

We can see that, in contrast with our previous work with a dualistic theory [16], for h there exist three non trivial solutions depending on the cosmological constant λ. But, at this preliminary analysis of the problem, only the values of h that make the quantity $(-h + \frac{1}{2}h^2) \in \mathbb{R}$ are relevant for our proposes: due the pure imaginary character of T in the Euclidean framework and mainly to compare with the NABI wormhole solution of our previous work (the question of the $h \in \mathbb{C}$ will be the focus of a further paper [9]). As the value of $h \in \mathbb{R}$ is -1 and in 4 spacetime dimensions $\lambda = |1-d| = 3$, then

$$T^a_{bc}|_{h_1} = \frac{3}{2}\frac{\varepsilon^a_{bc}}{a^2} \qquad T^a_{0c} = 0 \tag{51}$$

Namely, only the magnetic field is non vanishing while the electric field vanishes. An analogous feature can be seen in the solution of Giddings and Strominger [17] and in our previous paper [16]. Substituting the expression for the Torsion two form (51) into the symmetric part of the variational equation, namely[2]

$$R_{(\mu\nu)} = \overset{\circ}{R}_{\mu\nu} - T_{\mu\rho}{}^{\alpha} T_{\alpha\nu}{}^{\rho} = -2\lambda g_{\mu\nu} \tag{52}$$

[2]In the tetrad:

$$\overset{\circ}{R}_{00} = -3\frac{\ddot{a}}{a}, \qquad \overset{\circ}{R}_{ab} = -\left[\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 - \frac{2}{a^2}\right]$$

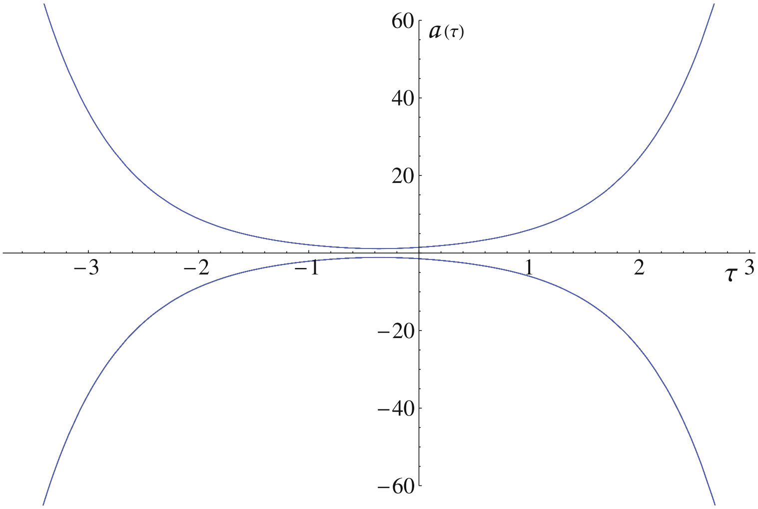

Fig. 1 Shape of the wormhole solution for values of the Euclidean time and torison $\tau_0 = 1$ and $T^a{}_{bc} = (3/2)\varepsilon^a{}_{bc}$, respectively

(in the tetrad: $\overset{\circ}{R}_{00} = -3\frac{\ddot{a}}{a}$, $\overset{\circ}{R}_{ab} = -\frac{1}{a}[\ddot{a}a + 2\dot{a}^2 - 2]$) we reduce (15) to an ordinary differential equation for the scale factor a,

$$\left[\left(\frac{\dot{a}}{a}\right)^2 - \frac{1}{a^2}\right] = \frac{2\lambda}{3} - \frac{9}{2a^4} \tag{53}$$

$$\frac{\mathrm{Ln}[1 + 4a^2 + 2\sqrt{-9 + 2a^2 + 4a^4}]}{2\sqrt{2}} = \tau - \tau_0 \tag{54}$$

$$\begin{aligned} T_{\mu\rho}{}^{\alpha}\, T_{\alpha\nu}{}^{\rho} &= \frac{(-h + \frac{1}{2}h^2)^2}{a^4} 2\delta_{\mu\nu} \\ &= \frac{9}{2a^4}\delta_{\mu\nu} \end{aligned} \tag{55}$$

There are 2 values for the scale factor a: max and min respectively, namely

$$a = \mp \frac{e^{-\sqrt{2}(\tau-\tau_0)}\sqrt{37 - 2e^{2\sqrt{2}(\tau-\tau_0)} + e^{4\sqrt{2}(\tau-\tau_0)}}}{2\sqrt{2}} \tag{56}$$

Expression (56) for the scale factor a is described in the Fig. 1 for the real value of h. As is easily seen from (56), the scale factor has an exponentially growing behavior, in sharp contrast to the wormhole solution from our previous work with the "dualistic" non-Abelian BI theory. Also, for this particular value of the torsion, the wormhole tunneling interpretation (in the sense of the Coleman' s mechanism) is fulfilled. Now will need to see what happens with (17) in this particular case under consideration. Well, (17) takes the following form

$$\begin{aligned} &d^*T^a + \frac{1}{2}\varepsilon^{abc}\left(f_b \wedge {}^*T_c - {}^*T_b \wedge f_c\right) = -2\lambda\, {}^*f^a \\ &(-2h + h^2)(1-h)d\tau \wedge e^b \wedge e^c = -2\lambda d\tau \wedge e^b \wedge e^c \end{aligned} \tag{57}$$

$$^*T^a \equiv h(-2h+h^2)d\tau \wedge \frac{e^a}{a^2} \tag{58}$$

$$^*f^a = -h\frac{d\tau \wedge e^b \wedge e^c}{a^3} \tag{59}$$

Then we arrived to the same equation for λ as (50) corroborating the self-consistency of the procedure.

6 Discussion and Concluding Remarks

In the previous section we shown that the non-dualistic unified model proposed here have deep differences with the dualistic non-Abelian Born-Infeld model of our early reference. The first obvious difference come from a conceptual framework: the geometrical action will provide, besides the spacetime structure, the matter-energy spin distribution. This fact is the same basis of the unification: all the (apparently disconnected) theories and interactions of the natural world appears naturally as a consequence of the intrinsic spacetime geometry. The second point to have account here is about the Hosoya and Ogura ansatz: why the identification of the isospin structure of the Yang-Mills field with the space frame lead a similar physical situation that a non-dualistic unified theory with torsion? The answer is: because at once such identification is implemented, a potential torsion is introduced and the solution of the set of equations is the consistency between the definition of the torsion tensor from the potential and the Cartan structure equations, namely

$$df = T + f^\alpha \wedge T^\beta \eta_{\alpha\beta} \tag{60}$$

$$D\omega^\alpha \equiv d\omega^\alpha + \omega^\alpha_\beta \wedge \omega^\beta = T^\alpha \tag{61}$$

$$\mathcal{R}^\alpha_\beta = D\omega^\alpha_\beta \tag{62}$$

Here, however, $f \equiv \frac{1}{2} f_{\alpha\beta}\omega^\alpha \wedge \omega^\beta$, $T \equiv T_{\alpha\beta\gamma}\omega^\alpha \wedge \omega^\beta \wedge \omega^\gamma$, $T^\alpha \equiv \frac{1}{2}T^\alpha_{\gamma\beta}\omega^\gamma \wedge \omega^\beta$ and $f^\alpha \equiv f^\alpha_\mu \omega^\mu$. The set of (57)–(58) is clearly self-consistent. The explanation from a pure algebraic and geometrical framework about the what happens with the underlying structure of the manifold is given with details in the Appendix.

The third point is about the obtained solutions for the scale factor, the difference with our previous work is precisely the particular form of the energy-momentum tensor in the NABI case (in the UFT model presented here, there are not energy-momentum tensor, of course): both solutions describe a wormhole-instanton but the final form of the differential equations for the scale factor are different, then the scale factor here has an exponentially growing behavior, in sharp contrast to the wormhole solution from our previous work with the "dualistic" non-Abelian BI theory. Also, for this particular value of the torsion, the wormhole tunneling interpretation (in the sense of the Coleman' s mechanism) is fulfilled.

The contact point between the compared models, however, are the dynamical equations that are very similar although the existence of a "current term" in the UFT model (cf. (45)) that not appears in the NABI case. This fact was pointed out in an slightly different context by N. Chernikov in [21].

The advantages of this nice model are clearly exposed in all this paper. The thinks to improve are:

(i) the dependence on the dimensions trough the cosmological constant $\lambda = |1-d|$
(ii) the lack of a manifest fermionic structure

(iii) tetrad-field depending on the breaking of symmetry of the underlying topological action, then the clear necessity of a reductive spacetime structure from the geometrical point of view.

Acknowledgements This paper is in the memory of the soviet scientist N.A. Chernikov, pioneering in remark the link between the Born-Infeld field equations and the general form of the affine geometries. Also many thanks are given to Professor Hubert F.M. Goenner for extremely useful discussions about the history of the gravitation and the concept of unification.

Appendix: On Coordinates and (An)-Holonomy

There is a confusion in the literature over the use of the word "coordinates." As a result, in the older literature influenced by J.A. Schouten [13], the terms "holonomic coordinate system" and "anholonomic system." are used. And for an anholonomic system an "anholonomic object" is employed. In the newer literature, exemplified by Bernard Schutz [18], the terms "coordinate system" and "noncoordinate system" are used. In this case the "anholonomic object" is replaced by the Lie algebra structure constant tensor. The key is to understand the relationships between manifolds and the vector fields which live on them. Also we must understand the difference between a commutative Lie group and a noncommutative Lie group and the effect which this difference makes on the vector fields on the respective Lie group manifolds. A coordinate system (= holonomic coordinate system) is characterized by the partial derivative nature of the vector fields associated with the coordinates. In symbols we can write that for coordinates, $x^1, x^2, \ldots,$ we have the vector field basis:

$$\frac{\partial}{\partial x^1} \quad \frac{\partial}{\partial x^2} \quad \cdots$$

Because of the fact that a partial derivative is with respect to one variable, and leaves all others fixed, the partial derivative operators are commutative. That is:

$$\left[\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}\right] \equiv \frac{\partial}{\partial x^1}\frac{\partial}{\partial x^2} - \frac{\partial}{\partial x^1}\frac{\partial}{\partial x^2} = 0$$

(the same for any x^i, x^j of course).

On any manifold, however, our starting point could be to consider the set of vector fields which live on the manifold. These vector fields are characterized by the flow lines (or integral curves) on the manifold. These flow lines can be used to describe coordinate systems on the manifold. In this case we will describe the vector fields in terms of the parameters along the flow lines. If we write these parameters with Greek letters μ, λ, etc. (to distinguish them from coordinates x^i), then we can write these vector fields as:

$$V = \frac{d}{d\mu} \qquad W = \frac{d}{d\lambda}$$

Notice that these are total differential operators. These operators are appropriate in case the operators do not commute. In this case the parameters, are not a parameterization appropriate to a coordinate system (or are "anholonomic coordinates" in the terminology of Schouten). As long as the vector fields V and W are independent, we can use them as a basis for a grid of parameters μ and λ. And, assuming V and W do not commute, this grid will not be a coordinate system (i.e. is "anholonomic"). Thus it is clear that the "anholonomic object" must be equivalent to the Lie bracket structure constants for a Lie algebra. For a Lie algebra this is a tensor. How then is it possible for the "anholonomic object" of a geometry to be coordinatized away? To understand this we need a simple example. Take the ordinary Euclidean plane R^2, with coordinates x and y. We can define the X and Y vector fields as:

$$X = \frac{\partial}{\partial x} \qquad Y = \frac{\partial}{\partial y}$$

This simply means that we fill up the x direction in the plane with a congruence of parallel flow lines for the vector field X, and similarly for the y direction. This is a perfectly commutative basis for R^2. However, we can also define polar "coordinates" (more correctly parameters) r and θ on R^2. In this case we can define the vector fields:

$$\widehat{r} = \cos\theta\ X + \sin\theta\ Y$$
$$\widehat{\theta} = -\sin\theta\ X + \cos\theta\ Y$$

and the commutator of these vector fields is:

$$\left[\widehat{r}, \widehat{\theta}\right] = -\frac{\widehat{\theta}}{r}$$

Thus $\widehat{r}$ and $\widehat{\theta}$ are a noncoordinate basis (cf. [18], p. 44). It is clear, however, that we can revert to a coordinate basis with X and Y as basis vector fields. So in this case the commutator "anholonomic" object can be coordinatized away by changing to the x, y axes as coordinates. This is possible because the underlying manifold R^2 is a commutative Lie group. Other examples of commutative Lie group manifolds are the n-dimensional vector spaces R^n, C^n of real or complex numbers and the n-dimensional torus T^n (i.e., a direct product of n circles S^1).

By now it should be clear that if the underlying manifold is a non-commutative Lie group, then the (non-commutative) Lie algebra of left-invariant vector fields on the Lie group manifold will provide a vector field basis (equivalent in dimensionality to that of the Lie group) which is a noncoordinate basis (i.e., "anholonomic"). And in this case the commutator of these vector fields is non-zero and thus the Lie algebra structure constant tensor is non-zero. This tensor plays the role of the "anholonomic object" and there is no way to coordinatize away this tensor. Moreover, the connection provided by the left-invariant vector fields provides an absolute parallelism structure on the Lie group manifold. (Note: absolute parallelism provides parallel transport of tangent vectors independent of the path throughout the Lie group manifold.)

This is connection is commonly called the Cartan connection because of his attempt to describe electromagnetism by way of the torsion tensor T associated with this asymmetric connection:

$$\Gamma^{\alpha}{}_{\beta\gamma} - \Gamma^{\alpha}{}_{\gamma\beta} = T^{\alpha}{}_{\beta\gamma}$$

This torsion tensor is equivalent to the Lie algebra structure constant tensor [12];

$$\left[X_i, X_j\right] = T^{k}{}_{ij} Z_k$$

(where T is usually written as C: the structure constant or function, in the general case).

In summary, three cases must clearly be distinguished:

(i) The underlying manifold is a commutative Lie group. (for example, R^n, C^n, T^n). In this case, the Lie algebra (of left-invariant vector fields) is commutative and thus provides a coordinate basis ("holonomic coordinates"). However, it is possible to set up a noncoordinate basis for vector fields, in which the basis fields do not commute. This sets up an artificial non-zero commutator, which plays the role of an "anholonomic object." But, clearly, it can be coordinatized away by reverting to the commutative Lie algebra basis structure of left-invariant vector fields. (Note that on any manifold there is an infinite dimensional basis of vector fields. However on a Lie group manifold, the action of the Lie group on itself and its vector fields provides for a finite set of left-invariant basis fields, where the dimensionality of this basis is that of the Lie group itself. This is the canonical basis for the Lie algebra of the Lie group.)

(ii) The underlying manifold is a noncommutative Lie group (for example, $SU(n)$, $SO(n)$, $E6$, $E7$, $E8$) In this case, the Lie algebra (of left-invariant vector fields) is noncommutative, and thus provides a noncoordinate basis ("anholonomic coordinates"). The Lie algebra structure constant tensor $C^k{}_{ij}$ plays many roles: "anholonomic object;" torsion tensor (relative to the Cartan connection); and (for particle physics) gauge group eigenvalues.

(iii) The underlying manifold is not a Lie group (for example, spheres S^n of any dimension n, except 1 and 3, since $S^1 = U(1)$, and $S^3 = SU(2)$ are Lie groups). This case may be of interest to certain applications of mechanics. However, it should be noted that according to the classification work of [19, 20], only Lie group manifolds are capable of carrying an absolute parallelism connection. The one exception to this rule is the 7-sphere S^7, which gets its parallelization from the fact that it is the set of unit length vectors in the 8-d Cayley algebra (the octonions).

Thus, if one is attempting to model electromagnetism via torsion in an absolute parallelism geometry, one should consider only the noncommutative Lie group case. (The commutative Lie groups carry no (Cartan type) torsion, of course).

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